

Actions of Tensor Categories and Cylinder Braids

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Abstract

Categorical actions of braided tensor categories are defined and shown to be the right framework for a discussion of the categorical structure related to the group of braids in the cylinder. A Kauffman polynomial of links in the solid torus is constructed.

1 Introduction

Braided tensor categories are the great unifying machine of braid and link theory. This paper introduces similar notions for braids in the cylinder and links in the solid torus.

Algebraically, the group of braids in the cylinder appears to be the braid group related to the Coxeter series B [1],[2],[11]. The generators $\tau_0, \tau_1, \dots, \tau_{n-1}$ obey

$$\tau_i \tau_j = \tau_j \tau_i \quad \text{if } |i - j| > 1 \quad (1)$$

$$\tau_i \tau_j \tau_i = \tau_i \tau_j \tau_i \quad \text{if } i, j \geq 1, |i - j| = 1 \quad (2)$$

$$\tau_0 \tau_i = \tau_i \tau_0 \quad \text{if } i \geq 2 \quad (3)$$

$$\tau_0 \tau_1 \tau_0 \tau_1 = \tau_1 \tau_0 \tau_1 \tau_0 \quad (4)$$

We denote this group by ZB_n . It may be graphically interpreted (cf. figure 1) as symmetric braids or cylinder braids: The symmetric picture shows it as the group of braids with $2n$ strands (numbered $-n, \dots, -1, 1, \dots, n$) which are fixed under a 180 degree rotation about the middle axis. In the cylinder picture one adds a single fixed line (indexed 0) on the left and obtains ZB_n as the group of braids with n strands that may surround this fixed line. The generators $\tau_i, i \geq 0$ are mapped to the corresponding diagrams given in figure 1.

More generally there are tangles (indicated in figure 1 by the TLJ tangles e_i) of B-type. They are used in the study of B-type Temperley-Lieb [1] and Birman-Wenzl [8] algebras.

The need for an extended theory of braided tensor categories arises because the braid generator τ_0 cannot be represented by a morphism in an ordinary braided tensor category. It does not satisfy the naturality condition with the A-type braiding τ_1 . We account for this fact by separating ordinary morphism which live in a braided tensor category from B-type morphisms which live in a non-tensor category that is a module over the braided tensor category. Graphically, the module action is given by putting the ordinary tangle to the right of a cylinder tangle. This setup has been suggested by tom Dieck [1], [3].

This generality is prompted by the desire to handle morphisms of the kind of e_0 in figure 1. Restricting to tangles that have only braidings around the cylinder one may do with a somewhat simpler concept introduced in [6].

The primary interest of the present paper lies in the formation of concepts. Proofs are rather sketchy, but may easily be enriched with more details. Physical applications that lurk in the background of this work may be found in [4], [5], [6].

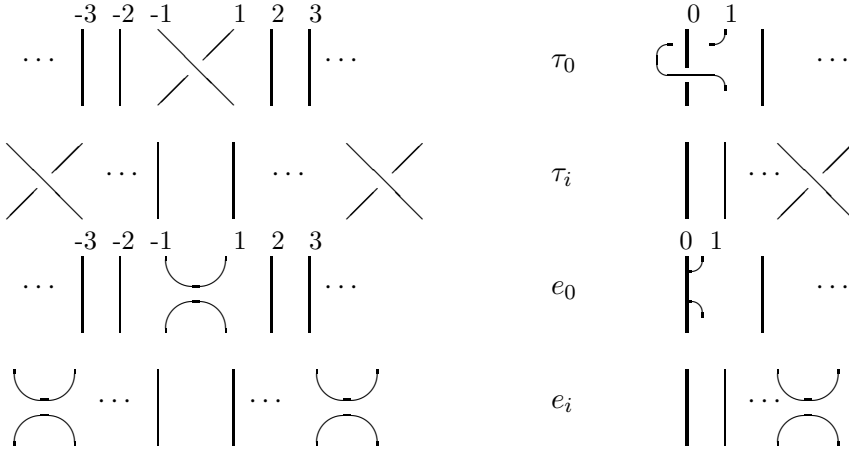


Figure 1: The graphical interpretation of the generators as symmetric tangles (on the left) and as cylinder tangles (on the right)

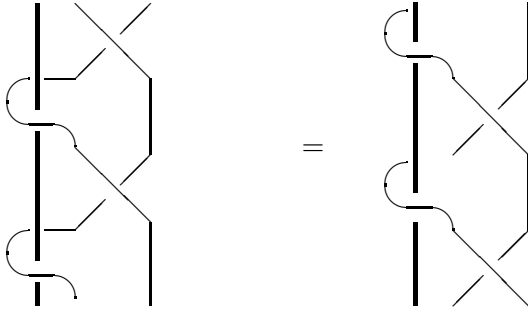


Figure 2: The cylinder interpretation of relation (4)

Tammo tom Dieck deserves thanks for discussions which stimulated much of the work of this paper.

Preliminaries: We use the notation of [9] for tensor categories. Expecially we denote by $a_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$ the associator and by $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ the braiding of a tensor category (resp. braided tensor category).

2 Actions of Tensor Categories

We formalise the notion of a tensor category acting on another category in the following way:

Definition 1 Let \mathcal{B} be a category and \mathcal{A} be a tensor category. We say that \mathcal{A} acts on \mathcal{B} (from the right) if there is a functor $*$: $\mathcal{B} \times \mathcal{A} \rightarrow \mathcal{B}$ such that the following axioms hold:

1. The following equation holds whenever both sides are defined:

$$(f * g)(f' * g') = (ff') * (gg') \quad (5)$$

2. There is a natural isomorphism $\lambda \in \text{Nat}(*(\text{Id} \times \otimes), *(* \times \text{Id}))$, i.e. $\lambda_{Y, X_1, X_2} : Y * X_1 \otimes X_2 \rightarrow Y * X_1 * X_2$ such that the following pentagon diagram commutes for all objects $Y \in \text{Obj}(\mathcal{B})$, $X_i \in \text{Obj}(\mathcal{A})$:

$$\begin{array}{ccc} Y * (X_1 \otimes X_2) \otimes X_3 & \xrightarrow{\text{id}_Y * a_{X_1, X_2, X_3}} & Y * X_1 \otimes (X_2 \otimes X_3) \\ & & \downarrow \lambda_{Y, X_1, X_2 \otimes X_3} \\ \downarrow \lambda_{Y, X_1 \otimes X_2, X_3} & & Y * X_1 * X_2 \otimes X_3 \\ & & \downarrow \lambda_{Y * X_1, X_2 \otimes X_3} \\ Y * (X_1 \otimes X_2) \otimes X_3 & \xrightarrow{\lambda_{Y, X_1, X_2} * \text{id}_{X_3}} & Y * X_1 * X_2 * X_3 \end{array} \quad (6)$$

3. There is a natural isomorphism $\rho_Y : Y * 1 \rightarrow Y$ such that

$$\begin{array}{ccc} Y * 1 \otimes X & \xrightarrow{\lambda_{Y, 1, X}} & Y * 1 * X \\ \downarrow \text{id}_Y * l_X & & \rho_Y * \text{id}_X \downarrow \\ Y * X & \xrightarrow{\text{id}_{Y * X}} & Y * X \end{array} \quad (7)$$

Here 1 denotes the unit object of \mathcal{A} and $l_X : 1 \otimes X \rightarrow X$ is its compatibility morphism in \mathcal{A} .

The pair $(\mathcal{B}, \mathcal{A})$ (together with the functor $*$) is called an action pair.

Examples 1 1. If $F : \mathcal{A} \rightarrow \mathcal{B}$ is a tensor functor between tensor categories then \mathcal{A} acts on \mathcal{B} by setting $X * Y := X \otimes F(Y)$, $\lambda_{X, Y_1, Y_2} := a_{X, F(Y_1), F(Y_2)}^{-1}$. As a special case any tensor category acts on itself.

2. Let \mathcal{A} be the category of bimodules over some ring R . This tensor category acts on the category \mathcal{B} of R right modules in the obvious way. This example is a special case of the former where the functor F is the forgetful functor from the category of bimodules to the category of right modules.

3. Let \mathcal{A} be a group considered as a tensor category, i.e. the objects are the group elements, tensor product is group multiplication. The endomorphism space of an object is some unital ring R while only one morphism $0 \in R$ exists between different objects. Assume that this group acts on a space \mathcal{B} which we consider as a category in a similar way. Then \mathcal{A} acts on \mathcal{B} in the sense of the above definition. This action is strict according to the definition given below.

Further examples will be given later on.

Definition 2 The action pair $(\mathcal{B}, \mathcal{A})$ is called strict if \mathcal{A} is a strict tensor category and one has $Y * X_1 * X_2 = Y * X_1 \otimes X_2$, $\lambda_{Y, X_1, X_2} = \text{id}_{Y * X_1 * X_2}$ and $\rho_Y = \text{id}_Y$.

Definition 3 Let $(\mathcal{B}, \mathcal{A})$ and $(\mathcal{B}', \mathcal{A}')$ be two action pairs. A functor between $(\mathcal{B}, \mathcal{A})$ and $(\mathcal{B}', \mathcal{A}')$ consists of:

1. A functor $F_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B}'$
2. A tensor functor $F_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}'$ with functorial morphisms φ_0, φ_2 defined as in [9][XI.4.1].
3. Natural isomorphisms $\omega_{X,Y} : F_{\mathcal{B}}(X * Y) \rightarrow F_{\mathcal{B}}(X) * F_{\mathcal{A}}(Y)$ such that the following diagram commutes

$$\begin{array}{ccc}
 F_{\mathcal{B}}(Y * X_1 \otimes X_2) & \xrightarrow{\lambda_{Y, X_1, X_2}} & F_{\mathcal{B}}(Y * X_1 * X_2) \\
 \downarrow \omega_{Y, X_1 \otimes X_2} & & \omega_{Y * X_1, X_2} \downarrow \\
 F_{\mathcal{B}}(Y) * F_{\mathcal{A}}(X_1 \otimes X_2) & & F_{\mathcal{B}}(Y * X_1) * F_{\mathcal{A}}(X_2) \\
 \downarrow \text{id} * \varphi_2(X_1, X_2)^{-1} & & \omega_{Y, X_1} * \text{id}_{F_{\mathcal{A}}(X_2)} \downarrow \\
 F_{\mathcal{B}}(Y) * F_{\mathcal{A}}(X_1) \otimes F_{\mathcal{A}}(X_2) & \xrightarrow{\lambda'_{F_{\mathcal{B}}(Y), F_{\mathcal{A}}(X_1), F_{\mathcal{A}}(X_2)}} & F_{\mathcal{B}}(Y) * F_{\mathcal{A}}(X_1) * F_{\mathcal{A}}(X_2)
 \end{array} \quad (8)$$

4. The following diagram commutes

$$\begin{array}{ccc}
 F_{\mathcal{B}}(Y * 1) & \xrightarrow{F_{\mathcal{B}}(\rho_Y)} & F_{\mathcal{B}}(Y) \\
 \downarrow \omega_{Y, 1} & & \rho'_{F_{\mathcal{B}}(Y)} \uparrow \\
 F_{\mathcal{B}}(Y) * F_{\mathcal{A}}(1) & \xrightarrow{\text{id} * \varphi_0^{-1}} & F_{\mathcal{B}}(Y) * 1
 \end{array} \quad (9)$$

Tensor categories can always be turned into strict ones by a procedure due to MacLane. A similar result holds in our situation:

Proposition 1 Every action pair $(\mathcal{B}, \mathcal{A})$ is equivalent to a strict action pair $(\mathcal{B}^{\text{str}}, \mathcal{A}^{\text{str}})$.

Proof. The proof is a variation of the proof of MacLanes's theorem. Hence we restrict ourselves to a sketchy description.

The objects of \mathcal{B}^{str} are sequences of one object of \mathcal{B} and arbitrary many objects from \mathcal{A} , i.e.

$$\begin{aligned}
 \text{Obj}(\mathcal{B}^{\text{str}}) &:= \{(Y, X_1, \dots, X_k) \mid Y \in \text{Obj}(\mathcal{B}), X_i \in \text{Obj}(\mathcal{A}), k \in \mathbb{N}_0\} \\
 \text{Obj}(\mathcal{A}^{\text{str}}) &:= \{(X_1, \dots, X_k) \mid X_i \in \text{Obj}(\mathcal{A}), k \in \mathbb{N}_0\}
 \end{aligned}$$

The equivalence functor is defined on objects by

$$\begin{aligned}
 F_{\mathcal{A}} &: \mathcal{A}^{\text{str}} \rightarrow \mathcal{A}, (X_1, \dots, X_k) \mapsto X_1 \otimes (X_2 \otimes (\dots)) \\
 F_{\mathcal{B}} &: \mathcal{B}^{\text{str}} \rightarrow \mathcal{B}, (Y, X_1, \dots, X_k) \mapsto Y * X_1 * \dots * X_k
 \end{aligned}$$

Morphism spaces are defined by

$$\begin{aligned}
 \text{Mor}_{\mathcal{A}}^{\text{str}}(S_1, S_2) &:= \text{Mor}_{\mathcal{A}}(F_{\mathcal{A}}(S_1), F_{\mathcal{A}}(S_2)) \\
 \text{Mor}_{\mathcal{B}}^{\text{str}}(S_1, S_2) &:= \text{Mor}_{\mathcal{B}}(F_{\mathcal{B}}(S_1), F_{\mathcal{B}}(S_2))
 \end{aligned}$$

The functors $F_{\mathcal{B}}, F_{\mathcal{A}}$ are essentially faithful and fully faithful. Hence, they are equivalences of categories. Their right inverses are defined by $Y \mapsto (Y)$.

Tensor product and action are defined by joining sequences. It remains to exhibit the natural isomorphism $\omega_{S,S'} : F_{\mathcal{B}}(S * S') \rightarrow F_{\mathcal{B}}(S) * F_{\mathcal{A}}(S')$. Its definition is recursive on the length of S' . One sets

$$\omega_{S,()}\coloneqq \rho_{F_{\mathcal{B}}(S)}^{-1}, \omega_{S,(X)}\coloneqq \text{id}, \omega_{S,(X)\otimes S'}\coloneqq \lambda_{F_{\mathcal{B}}(S),X,F_{\mathcal{A}}(S')}^{-1}\omega_{S*(X),S'}$$

The key lemma to establish (8) is

Lemma 2 $\lambda_{F_{\mathcal{B}}(S),F_{\mathcal{A}}(S'),F_{\mathcal{A}}(S'')}(\text{id}_{F_{\mathcal{B}}(S)} * \varphi_2(S',S'')^{-1})\omega_{S,S'\otimes S''} = (\omega_{S,S'} * \text{id}_{F_{\mathcal{A}}(S'')})\omega_{S*S',S''}$

It is shown by induction on the length of S' . \square

The strictification of action pairs simplifies considerably the task of specifying them by generators and relations in a fashion similar to the presentation of braided tensor categories given in [9][XII.1]. One starts with a strict action pair $(\mathcal{B}, \mathcal{A})$ and singles out a set $\mathcal{F}_{\mathcal{B}}$ of morphisms from \mathcal{B} . They are used to build formal words defined recursively by their length: Words of length 1 are $[f]$ where $f \in \mathcal{F}$ and $[\text{id}_Y]$, $Y \in \text{Obj}(\mathcal{B})$. If a, b are words of length $\leq n$ and g is a morphism from \mathcal{A} then $a * g$ and ab are words of length $n + 1$. To every word a morphism of \mathcal{B} is associated by the rules $\overline{[f]} := f, \overline{a * g} := \overline{a} * g, \overline{ab} := \overline{a} \circ \overline{b}$. The set of sub-words of a word is also defined recursively by $\text{sub}([f]) := \{[f]\}$, $\text{sub}(a * b) := \{b\} \cup \text{sub}(a)$, $\text{sub}(ab) := \text{sub}(a) \cup \text{sub}(b)$. Two words a, b are said to be equivalent $a \sim b$ iff there exists a sequence of words a_i with $a_0 = a, a_k = b$ and a_{i+1} is obtained from a_i by one of the following transformations applied to a sub-word: $(ab)c \sim a(bc), [\text{id}]a \sim a, a[\text{id}] \sim a, a * \text{id}_1 \sim a, [\text{id}_Y * X] \sim [\text{id}_Y] * \text{id}_X, a * gg' \sim a * g * g', (a * g)(a' * g') \sim (aa') * (gg')$. From this one concludes that $(a * \text{id}_{b(g)})([\text{id}_{s(\overline{a})}] * g) \sim ([\text{id}_{b(\overline{a})}] * g)(a * \text{id}_{s(g)})$ and $(a_1 * \text{id}) \cdots (a_k * \text{id}) \sim (a_1 \cdots a_k) * \text{id}$. A simple inductive proof shows that any word is equivalent to one of the form $h_1 \cdots h_m$ where each h_i is of the form $[f] * \text{id}_X$ with $f \in \mathcal{F}$ or of the form $[\text{id}_X] * g$.

The free action pair generated by \mathcal{F} is the pair $(\mathcal{M}(\mathcal{F}), \mathcal{A})$ where $\mathcal{M}(\mathcal{F})$ has the same objects as \mathcal{B} but its morphism space is the set of equivalence classes of words.

Further relations $\mathcal{R} = \{(r_i, r'_i) \mid i = 1..k\}$ can be used to define another equivalence relation $a \sim_{\mathcal{R}} b$ on words where one may also replace a sub-word r_i by r'_i or vice versa. One then says that the action pair $(\mathcal{B}, \mathcal{A})$ is generated by \mathcal{F} with relations \mathcal{R} if every morphism of \mathcal{B} can be obtained as \overline{a} from a word and one has $a \sim_{\mathcal{R}} b \Leftrightarrow \overline{a} = \overline{b}$.

3 Cylinder twists

This section introduces the cylinder braid morphism.

Definition 4 *A strict action pair $(\mathcal{B}, \mathcal{A})$ is said to be cylinder braided if:*

1. $\text{Obj}(\mathcal{B}) = \text{Obj}(\mathcal{A})$ and $1 * X = X$
2. \mathcal{A} is a braided tensor category with braid isomorphisms $c_{X,Y} \in \text{Mor}_{\mathcal{A}}(X \otimes Y, Y \otimes X)$
3. For every object there exists an isomorphism $t_X \in \text{Mor}_{\mathcal{B}}(X, X)$ such that

$$\begin{aligned} c_{Y,X}(t_Y \otimes \text{id}_X)c_{X,Y}(t_X \otimes \text{id}_Y) &= (t_X \otimes \text{id}_Y)c_{Y,X}(t_Y \otimes \text{id}_X)c_{X,Y} = t_{X \otimes Y} \quad (10) \\ ft_X &= t_Y f \quad \forall f \in \text{Mor}_{\mathcal{A}}(X, Y) \quad (11) \end{aligned}$$

4. The following equations should hold if \mathcal{A} is equipped with a duality.

$$(t_X \otimes \text{id}_{X^*})b_X = c_{X,X^*}^{-1}(t_{X^*}^{-1} \otimes \text{id}_X)c_{X^*,X}^{-1}b_X \quad (12)$$

$$d_X(t_X^{*-1} \otimes \text{id}_X) = d_X c_{X,X^*}(t_X \otimes \text{id}_{X^*})c_{X^*,X} \quad (13)$$

t is called the cylinder twist. For the sake of brevity we call $(\mathcal{B}, \mathcal{A})$ (or even \mathcal{B}) a cylinder braided tensor category CBTC.

The requirements of strictness and those of point 1 of the definition imply that $X * Y = 1 * X * Y = 1 * X \otimes Y = X \otimes Y$. Note also that in the light of (10) relations (12), (13) may be rewritten as $t_{X \otimes X^*} b_X = b_X$ and $d_X t_{X^* \otimes X} = d_X$.

Remark 1 1. The space $\text{End}_{\mathcal{B}}(X^{\otimes n})$ carries a representation of the braid group ZB_n .

2. Assume there are m distinct morphisms $t^{(1)}, \dots, t^{(m)}$ such that each product of pairwise different $t^{(i)}$ makes the action pair $(\mathcal{B}, \mathcal{A})$ cylinder braided. Then one has a representation of the braid group of the handlebody of genus m [12].
3. Our action pairs are defined by a right action of \mathcal{A} . Similarly one can consider left actions. Suppose \mathcal{A} to act on \mathcal{B} from the right and on \mathcal{B} from the left. If both of these actions are cylinder braided then one has a tensor representation of the braid group of the affine Coxeter diagram $\bullet = \bullet - \bullet - \dots - \bullet - \bullet = \bullet$.

The fundamental geometric example of tangles in the cylinder will be described in the next section. Here we restrict ourselves to some simple examples.

Examples 2 1. A ribbon category \mathcal{A} acting on itself is trivially a cylinder braided pair where the cylinder twist is given by the ribbon twist $t_X = \theta_X$.

2. An abelian group G together with bilinear pairing $c : G \times G \rightarrow K^*$ with values in the group of units of a commutative unital ring K may be viewed as a braided tensor category \mathcal{A} as in [13][p. 29]. The pair $(\mathcal{A}, \mathcal{A})$ is then cylinder braided if there is a map $t : G \rightarrow K^*$ such that $t(gg') = c(g, g')c(g', g)t(g)t(g')$. In the symmetric case t is simply a group character.
3. Let \mathcal{A} be a tensor category and $X \in \text{Obj}(\mathcal{A})$ any object. This category acts on \mathcal{B} which has the same objects and morphisms $\text{Mor}_{\mathcal{B}}(X_1, X_2) := \text{Mor}(X \otimes X_1, X \otimes X_2)$. The action is given by the monoidal product and cylinder twist is $t_Y := c_{Y,X}c_{X,Y}$.

Further examples are provided by the Coxeter-B braided tensor categories studied in [6]. That paper contains a discussion of cylinder braid structures on Hopf algebras and Tannaka-Krein duality.

4 Cylinder Ribbon Tangles

The fundamental example of a cylinder braided action pair is the pair $(\text{CylRib}, \text{Rib})$. Rib is Turaev's category of ribbon tangles and CylRib is the category of cylinder ribbon tangles which is defined just like Rib but with the restriction that the tangles extend only in the

space $(\mathbb{R}^2 - (0,0)) \times [0,1]$. The action of a tangle f from Rib on a tangle g from CylRib is given by putting f to the right of g . The category of \mathcal{A} coloured cylinder ribbon tangles $\text{CylRib}_{\mathcal{A}}$ parallels the category $\text{Rib}_{\mathcal{A}}$.

We use Turaev's notation for the generators of Rib. The basic generators of CylRib are $\tau^{\downarrow\pm}\tau^{\uparrow\pm}$. They are oriented versions of τ_0 given in figure 1 and its inverse. The arrow indicates the orientation. The lines are meant to represent ribbons that with the framing oriented towards the axis.

Proposition 3 *The following list of relations holds in CylRib:*

$$\tau^{\downarrow+} = \tau^{\downarrow-}{}^{-1} \quad (14)$$

$$\tau^{\uparrow+} = \tau^{\uparrow-}{}^{-1} \quad (15)$$

$$\tau^{\downarrow-} = (\cap \otimes \downarrow)(\varphi'^{\uparrow} \otimes \downarrow \otimes \downarrow)(\tau^{\uparrow+} \otimes X^-)(\cup^- \otimes \downarrow) \quad (16)$$

$$\tau^{\uparrow-} = (\cap^- \otimes \uparrow)(\varphi'^{\downarrow} \otimes \uparrow \otimes \uparrow)(\tau^{\downarrow+} \otimes T^-)(\cup \otimes \uparrow) \quad (17)$$

$$(\tau^{\downarrow+} \otimes \downarrow)X^+(\tau^{\downarrow+} \otimes \downarrow)X^+ = X^+(\tau^{\downarrow+} \otimes \downarrow)X^+(\tau^{\downarrow+} \otimes \downarrow) \quad (18)$$

$$(\tau^{\uparrow+} \otimes \uparrow)T^+(\tau^{\uparrow+} \otimes \uparrow)T^+ = T^+(\tau^{\uparrow+} \otimes \uparrow)T^+(\tau^{\uparrow+} \otimes \uparrow) \quad (19)$$

$$(\tau^{\downarrow+} \otimes \uparrow)Y^-(\tau^{\uparrow+} \otimes \downarrow)Z^- = Y^-(\tau^{\uparrow+} \otimes \downarrow)Z^-(\tau^{\downarrow+} \otimes \uparrow) \quad (20)$$

$$(\tau^{\uparrow+} \otimes \downarrow)Z^-(\tau^{\downarrow+} \otimes \uparrow)Y^- = Z^-(\tau^{\downarrow+} \otimes \uparrow)Y^-(\tau^{\uparrow+} \otimes \downarrow) \quad (21)$$

$$(\tau^{\downarrow+} \otimes \uparrow)\cup = Y^+(\tau^{\uparrow-}\varphi^{\uparrow} \otimes \downarrow)\cup^- \quad (22)$$

$$\cap(\tau^{\uparrow-} \otimes \downarrow) = \cap^-(\tau^{\downarrow+}\varphi'^{\downarrow} \otimes \downarrow)Y^- \quad (23)$$

$$(\uparrow \otimes \varphi)\cup^- = (\tau^{\uparrow+} \otimes \downarrow)Z^-(\tau^{\downarrow+} \otimes \uparrow)\cup \quad (24)$$

$$\cap(\uparrow \otimes \varphi) = \cap^-(\tau^{\downarrow+} \otimes \uparrow)Y^-(\tau^{\uparrow+} \otimes \downarrow) \quad (25)$$

$$(\downarrow \otimes \varphi^{\uparrow})\cup = (\tau^{\downarrow+} \otimes \uparrow)Y^-(\tau^{\uparrow+} \otimes \downarrow)\cup^- \quad (26)$$

$$\cap^-(\downarrow \otimes \varphi^{\uparrow}) = \cap(\tau^{\uparrow+} \otimes \downarrow)Z^-(\tau^{\downarrow+} \otimes \uparrow) \quad (27)$$

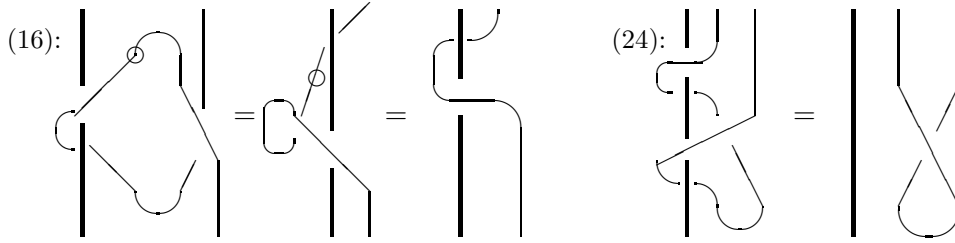


Figure 3: Two of the relations of CylRib. The small circle denotes a ribbon twist φ' .

The proof is a simple verification. Some of the pictorial calculations are given in figure 3. Because of (16) and (17) only $\tau^{\downarrow+}, \tau^{\uparrow+}$ are needed as generators. This reduces the numbers of relations because (16) and (17) turn (22) and (23) into identities that involve only Rib operations.

Proposition 4 *The set $\mathcal{F} := \{\tau^{\downarrow+}, \tau^{\uparrow+}\}$ generates the action pair $(\text{CylRib}, \text{Rib})$ with relations (18)-(21), (24)-(27).*

Proof. Tangles in the cylinder may be interpreted as ordinary tangles with a fixed additional strand. The question of equivalence of diagrams can thus be reduced to the situation

in \mathbb{R}^3 [13]. However, ordinary Markov moves may easily produce diagrams that are no longer products of our generators. We therefore need a method to produce a standard form (a product of generators) from an arbitrary diagram. There are several such methods. We use the R-process. It is based on horizontal diagrams, i.e. regular projections of cylinder links on a horizontal plane. In contrast we call the diagrams used so far standard diagrams. In a horizontal diagram the cylinder axis is just a point. To avoid upper and lower end points from being projected on the same point we shift them in opposite directions parallel to the second coordinate axis. Upon multiplication we have to join such endings with horizontal ribbons. Figure 4 displays an example.

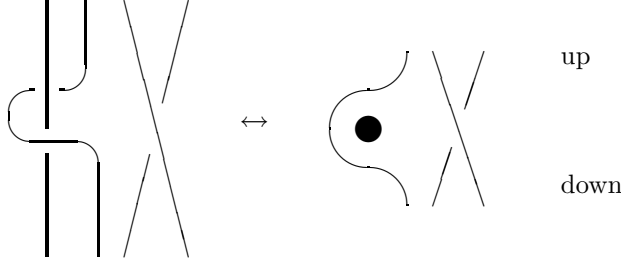


Figure 4: A simple example of a horizontal diagram

Let such a horizontal diagram be given and choose a line (the radar beam) from the point of the axis and extending into the left half plane such that it hits the ribbons transversal and avoids crossings. The tangle is then deformed away from the beam until all of its nontrivial part is located in the right half plane as indicated in figure 5. We may assign a standard diagram to the result by drawing a sequence of τ morphisms for every circle surrounding the axis such that the innermost circle corresponds to the lowest τ . Diagrams that differ by some kind of Reidemeister move that takes place either above or below the radar beam are transformed to standard diagrams that are related by precisely the same move at a different position. It remains to discuss how the result depends on the choice of the radar beam. Essentially, there are only two relevant possibilities that correspond to Reidemeister moves of types II and III. We concentrate on the type III move. Consider two situations differing only by the position of a single crossing with respect to the radar beam. The beam may either be above or below the crossing. Figure 6 shows diagrams of these situations. We demand that the tangle in the right half space is concentrated in a diagonal box so that the connection points with the axis surrounding circles are projected both horizontally and vertically to the same order. Then it is easy to determine how the parts fit together. Comparing diagrams on the right of figure 6 yields the four braid relation. A similar argument using a minimum (maximum) lying above or below the radar beam yields the extremum twist relation. Putting in orientations these relations are precisely (18)-(21), (24)-(27). \square

Proposition 5 *There is a unique tensor functor between strict action pairs $F : (\text{CylRib}_{\mathcal{A}}, \text{Rib}_{\mathcal{A}}) \rightarrow (\mathcal{B}, \mathcal{A})$ such that $F_{\mathcal{A}}$ is Turaev's functor, the functorial isomorphism ω is trivial and one has*

$$F_{\mathcal{B}}(\tau_X^{\downarrow \pm}) = t_X^{\pm 1} \quad (28)$$

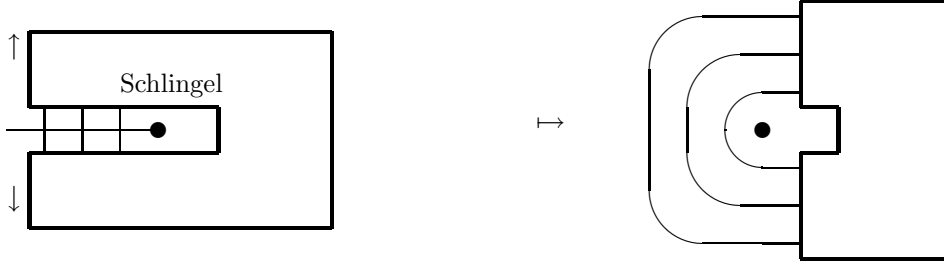


Figure 5: Deforming a horizontal diagram: On the left the original diagram with a radar beam and arrows indicating the direction of deformation. The result is shown on the right.

$$F_{\mathcal{B}}(\tau_X^{\pm}) = t_X^{\pm 1} \quad (29)$$

Proof. Uniqueness is clear because $F_{\mathcal{B}}$ is fixed on generators. To prove existence one has to check compatibility with the relations given above. This is done by straightforward graphical computations which are however too long to be displayed here. \square

5 Cylinder braided action pairs with Points

Until now we have no possibility to represent the diagram e_0 of figure 1 which plays a crucial role in the study of some B-type knot algebras. The point structure discussed in this section fills the gap.

Definition 5 *A point structure on a CBTC $(\mathcal{B}, \mathcal{A})$ (where \mathcal{A} is rigid) consists of a point morphism $\bar{b}_X \in \text{Mor}_{\mathcal{B}}(1, X)$ and copoint morphisms $\bar{d}_X \in \text{Mor}_{\mathcal{B}}(X, 1)$ such that the following axioms are fulfilled.*

$$d_Y^0 f = d_X^0 \quad f b_X^0 = b_Y^0 \quad \forall f \in \text{Mor}_{\mathcal{A}}(X, Y) \quad (30)$$

$$d_X(\bar{b}_{X^*} \otimes \text{id}_X) = \bar{d}_X \quad (31)$$

$$(\bar{d}_X \otimes \text{id}_{X^*}) b_X = \bar{b}_{X^*} \quad (32)$$

$$\bar{b}_{X \otimes Y} = (\bar{b}_X \otimes \text{id}_Y) \bar{b}_Y \quad (33)$$

$$\bar{d}_{X \otimes Y} = \bar{d}_Y (\bar{d}_X \otimes \text{id}_Y) \quad (34)$$

$$\bar{b}_X = t_X \bar{b}_X \quad (35)$$

$$\bar{d}_X = \bar{d}_X t_X \quad (36)$$

$$(t_Y \otimes \text{id}_X) c_{X,Y} (\bar{b}_X \otimes \text{id}_Y) = c_{Y,X}^{-1} (\bar{b}_X \otimes \text{id}_Y) t_Y \quad (37)$$

$$t_Y (\bar{d}_X \otimes \text{id}_Y) = \bar{d}_X c_{Y,X} (t_Y \otimes \text{id}_X) c_{X,Y} \quad (38)$$

Some simple consequences are:

$$\bar{d}_{X^*} = d_X c_{X,X^*} (\theta_X \bar{b}_X \otimes \text{id}_{X^*}) \quad (39)$$

$$\bar{b}_{X^*} = (\bar{d}_{X^*} \otimes \text{id}_X) (\text{id}_{X^*} \otimes \theta_X^{-1}) c_{X^*,X}^{-1} b_X \quad (40)$$

A point structure is the B-type analog of duality (rigidity).

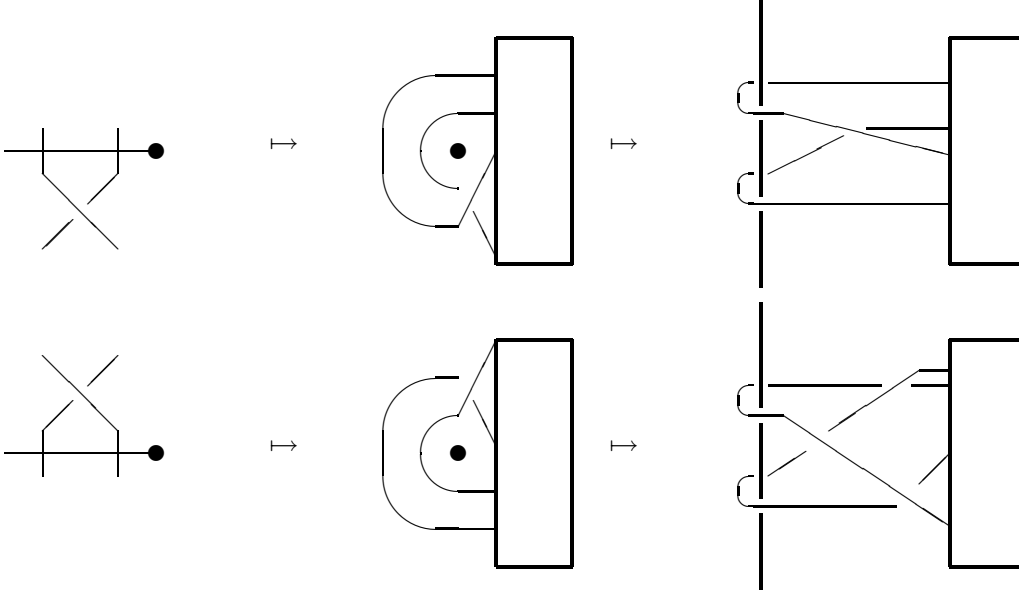


Figure 6: The pictures in the upper and lower row differ only by the position of a single crossing relative to the chosen radar beam. Irrelevant parts of the diagram are omitted. The second mapping associates a standard diagram to the horizontal diagram.

$(\text{CylRib}, \text{Rib})$ has no point structure. We define $(\text{PCylRib}, \text{Rib})$ as an extension where ribbons are allowed to end at the cylinder axis. Figure 7 display the point and copoint morphisms. Note that points (i.e endings of ribbons on the axis) do not commute, i.e there is no way to simplify the picture on the right of figure 7.

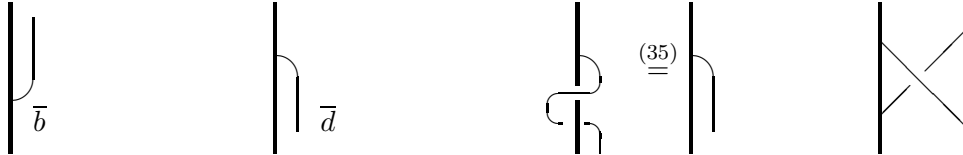


Figure 7: Point and copoint of PCylRib

6 Skein relations

In PCylRib one can impose skein relations that generalise those of the Kauffman polynomial:

$$c - c^{-1} = \delta(1 - bd) \quad (41)$$

$$cb = \lambda b \quad dc = \lambda d \quad (42)$$

$$db = A_0 \quad (43)$$

$$t^{-1} = \alpha t + \beta + \gamma b^0 d^0 \quad (44)$$

$$d^0 b^0 = x_0 \quad (45)$$

$$db^0 d^0 b = x'_0 \quad (46)$$

$$d(t \otimes \text{id})b = A_1 \quad (47)$$

$$d(t^{-1} \otimes \text{id})b = A_{-1} \quad (48)$$

$$(d^0 \otimes \text{id})c(b^0 \otimes \text{id}) = \epsilon + \mu t + \nu b^0 d^0 \quad (49)$$

The parameters are $\delta, A_0, \lambda, \alpha, \beta, \gamma, A_1, A_{-1}, x_0, x'_0, \epsilon, \mu, \nu$.

Assuming that the annihilator ideals of the generators vanish we can derive a set of relations between these parameters. As in the case of the A-type category of the usual Kauffman polynomial one has

$$A_0 \delta - \delta = \lambda - \lambda^{-1}$$

We have $d^0 = d^0 t^{-1} = \alpha d^0 + \beta d^0 + \gamma d^0 b^0 d^0 = (\alpha + \beta + \gamma x_0) d^0$. and hence:

$$1 = \alpha + \beta + \gamma x_0$$

Similarly:

$$A_{-1} = \alpha A_1 + \beta A_0 + \gamma x'_0$$

Multiplying $\lambda^{-1}(t^{-1} \otimes \text{id})b = c(t \otimes \text{id})b$ with d we obtain

$$A_1 \lambda^2 = A_{-1}$$

Next, we calculate $\gamma x_0 b^0 d^0 = \gamma b^0 d^0 b^0 d^0 = b^0 d^0 (t^{-1} - \alpha t - \beta) = b^0 d^0 (1 - \alpha - \beta)$ and obtain

$$\gamma x_0 = 1 - \alpha - \beta$$

Similarly $x'_0 = db^0 d^0 b = \gamma^{-1}(d(t^{-1} \otimes 1)b - \alpha d(t \otimes 1)b - \beta db)$:

$$\gamma x'_0 = A_{-1} - \alpha A_1 - \beta A_0$$

Finally, tensor (44) with c and multiply with $d \otimes \text{id}$ from the left and with $b \otimes \text{id}$ from the right. The result may be brought to a form which resembles (49). Comparing coefficients one obtains:

$$\begin{aligned} \nu &= -\alpha \lambda \\ \mu &= \gamma^{-1}(\alpha \delta - \alpha^2 \lambda + \lambda^{-1}) \\ \epsilon &= -\gamma^{-1}(\alpha \beta \lambda + \alpha \delta A_1 + \beta \lambda^{-1}) \end{aligned}$$

Only 4 of 13 parameters survive. We may reduce this number even more, if we demand that $x_0 = x'_0$.

Link in the solid torus are endomorphisms of the 0-object. Kauffman's Theory [10] can be used to eliminate ordinary braidings c . The remaining tangles can be simplified using (49). Therefore the skein relations suffice to calculate a cylinder generalisation of Kauffman's polynomial.

Just as with Kauffman's original polynomial this link invariant may also be obtained as a writhe normalisation of a Markov trace on a B-type generalisation of Birman-Wenzl algebra [7].

For special values of the parameters it may also be derived from tensor representations of $PCylRib$. Upto now only two nontrivial tensor representations have been found. They are based on tensor representations of the B-type braid group that use R-matrices of the orthogonal quantum groups (found by tom Dieck [2] for $\mathcal{U}_{q\mathfrak{so}_3}$ and by myself for $\mathcal{U}_{q\mathfrak{so}_5}$ (unpublished)).

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